# CONNECTIVITY PROPERTIES OF STAR AND STAR/MESH NETWORKS

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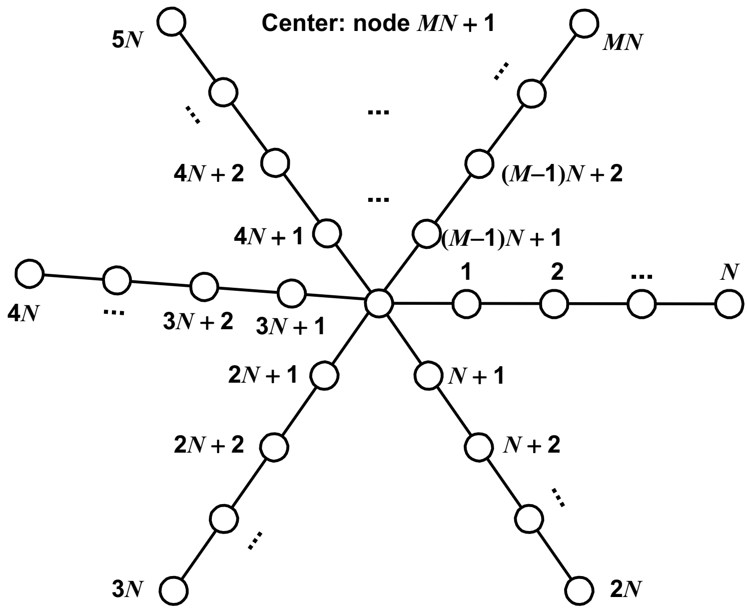
## 1. STAR NETWORK

A star network connected by bidirectional links can be modeled as $M$ "rays" of $N$ nodes plus a center node, as illustrated in the following figure:




The adjacency (one-hop connectivity) matrix for such a network, in which a 1 entry at $(i, j)$ indicates a connection from node $i$ to node $j$ and a 0 entry at $(i,\ j)$ indicates no connection from node $i$ to node $j$ , is an $(NM+1)\times(NM+1)$ matrix with the form given by

$$A_{star} = \left[\begin{array}{cccccc} A_N & 0 & 0 & \cdots & 0 & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ 0 & A_N & 0 & \cdots & 0 & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ 0 & 0 & A_N & \cdots & 0 & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_N & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \cdots & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & 0 \end{array}\right] \tag{1}$$

where $A_N$ is the $N \times N$ adjacency matrix for a single row of $N$ nodes connected in tandem. The structure of $A_N$ is a matrix of 0s, except for $N-1$ 1s on the first upper diagonal and $N-1$ 1s on the first lower diagonal, for example,

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

The one-hop connectivity of the network, defined as the fraction of the $NM(NM+1)$ possible links that are operative, is simply the sum of the elements of $A_{star}$ divided by $NM(NM+1)$, or

$$\textit{Connectivity} = \frac{M \times 2(N-1) + 2M \times 1}{NM(NM+1)} = \frac{2}{NM+1} \tag{3}$$

In addition to one-hop connectivity, we are interested in the hop distance between each pair of nodes, defined as the minimum number of links needed to be traversed in order to connect the pair. The hop distances for the possible node pairs can be represented collectively as entries in a (multihop) connectivity matrix. With some observation, it can be verified that the (multihop) connectivity matrix for the star network with $M$ rays of $N$ nodes plus a center node has the form given by

$$C_{star} = \begin{bmatrix} C_N & C_N + 2V_N & C_N + 2V_N & \cdots & C_N + 2V_N & \boldsymbol{v}_N \\ C_N + 2V_N & C_N & C_N + 2V_N & \cdots & C_N + 2V_N & \boldsymbol{v}_N \\ C_N + 2V_N & C_N + 2V_N & C_N & \cdots & C_N + 2V_N & \boldsymbol{v}_N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_N + 2V_N & C_N + 2V_N & C_N + 2V_N & \cdots & C_N & \boldsymbol{v}_N \\ \boldsymbol{v}_N^T & \boldsymbol{v}_N^T & \boldsymbol{v}_N^T & \cdots & \boldsymbol{v}_N^T & 0 \end{bmatrix} \tag{4}$$

where $C_N$ is the connectivity matrix for a row of $N$ nodes, $\boldsymbol{v}_N^T = (1, 2, 3, \cdots, N)$ is a $1 \times N$ vector (the transpose of $\boldsymbol{v}_N$), and $V_N$ is a special $N \times N$ matrix given by

$$V_N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 2 & 3 & 4 & 4 & \cdots & 4 \\ 1 & 2 & 3 & 4 & 5 & \cdots & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & N \end{bmatrix} \tag{5}$$

The connectivity matrix for a row of $N$ nodes is given by

$$C_N = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & N-1 \\ 1 & 0 & 1 & 2 & 3 & \cdots & N-2 \\ 2 & 1 & 0 & 1 & 2 & \cdots & N-3 \\ 3 & 2 & 1 & 0 & 1 & \cdots & N-4 \\ 4 & 3 & 2 & 1 & 0 & \cdots & N-5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N-1 & N-2 & N-3 & N-4 & N-5 & \cdots & 0 \end{bmatrix} \tag{6}$$

Note that the elements on the $k$th upper and lower diagonals of $C_N$ are all equal to $k$, $k = 1, 2, \ldots, N-1$. For example, the connectivity matrix for a star network with three rays of four nodes plus a center node is the following $13 \times 13$ matrix:

$$C_{star} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 2 \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & 3 \\ \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 & 4 \\ 2 & 3 & 4 & 5 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 6 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 3 & 4 & 5 & 6 & 2 \\ 4 & 5 & 6 & 7 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 4 & 5 & 6 & 7 & 3 \\ 5 & 6 & 7 & 8 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 5 & 6 & 7 & 8 & 4 \\ 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 \\ 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 \\ 4 & 5 & 6 & 7 & 4 & 5 & 6 & 7 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 \\ 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 0 \end{bmatrix} \tag{7}$$

The maximum hop distance between node pairs is $2N$. The average hop distance is simply the sum of the elements of $C_{star}$ divided by $NM(NM+1)$. To derive the average hop distance for the star network, let us use the notation $\|X\|_+$ to denote the sum of all the elements of matrix $X$. Then the average hop distance is given by

$$\overline{m} = \frac{\|C_{star}\|_+}{NM(NM+1)} \tag{8}$$

By inspection of (4), the sum of the elements of $C_{star}$ is given by

$$\|C_{star}\|_+ = M^2\|C_N\|_+ + 2M\|\boldsymbol{v}_N\|_+ + 2M(M-1)\|V_N\|_+ \tag{9}$$

where $\|\boldsymbol{v}_N\|_+ = \frac{1}{2}N(N+1)$,

$$\begin{aligned} \|C_N\|_+ &= 2[(N-1)\cdot 1 + (N-2)\cdot 2 + \cdots + 2\cdot(N-2) + 1\cdot(N-1)] \\ &= 2\sum_{k=1}^{N-1}(N-k)k = 2N\sum_{k=1}^{N-1}k - 2\sum_{k=1}^{N-1}k^2 \end{aligned}$$

$$= 2N \cdot \tfrac{1}{2}(N-1)N - 2 \cdot \tfrac{1}{6}(N-1)N(2N-1) = N(N-1)\left(\tfrac{N+1}{3}\right) \tag{10}$$

and

$$\begin{aligned}
\|V_N\|_+ &= 1 \cdot (2N-1) + 2 \cdot (2N-3) + \cdots + (N-1) \cdot 3 + N \cdot 1 \\
&= \sum_{k=1}^{N} k(2N-2k+1) = (2N+1)\sum_{k=1}^{N} k - 2\sum_{k=1}^{N} k^2 \\
&= (2N+1) \cdot \tfrac{1}{2}N(N+1) - 2 \cdot \tfrac{1}{6}N(N+1)(2N+1) \\
&= \tfrac{1}{6}N(N+1)(2N+1)
\end{aligned} \tag{11}$$

Thus

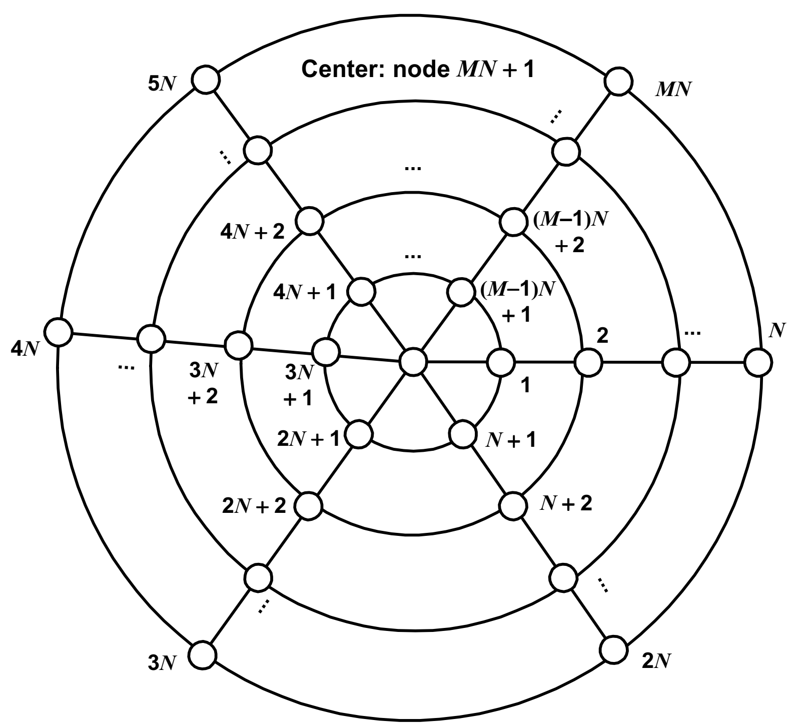
$$\begin{aligned}
\|C_{star}\|_+ &= M^2 \cdot N(N-1)\left(\tfrac{N+1}{3}\right) + MN(N+1) + M(M-1) \cdot \tfrac{1}{3}N(N+1)(2N+1) \\
&= \tfrac{1}{3}NM(N+1)(2MN+M-N+1)
\end{aligned} \tag{12}$$

and the average hop distance for a star network is

$$\overline{m} = \frac{(N+1)(2MN+M-N+1)}{3(MN+1)} \tag{13}$$

2. STAR-MESH NETWORK

If the nodes in the star network are cross-connected as well as radially connected, then a kind of "star-mesh" network topology is created, which has the following diagram:



The adjacency (one-hop connectivity) matrix for such a network, in which a 1 entry at $(i, j)$ indicates a connection from node $i$ to node $j$ and a 0 entry at $(i,\ j)$ indicates no connection from node $i$ to node $j$ , is an $(NM+1) \times (NM+1)$ matrix with the form given by

$$A_{star\text{-}mesh} = \begin{bmatrix} A_N & I_N & 0 & \cdots & I_N & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ I_N & A_N & I_N & \cdots & 0 & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ 0 & I_N & A_N & \cdots & 0 & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ I_N & 0 & 0 & \cdots & A_N & \begin{smallmatrix}1\\0\\\vdots\\0\end{smallmatrix} \\ \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & \cdots & \begin{smallmatrix}1 & 0 & \cdots & 0\end{smallmatrix} & 0 \end{bmatrix} \tag{14}$$

where $I_N$ is the $N \times N$ identity matrix and $A_N$ is the $N \times N$ adjacency matrix for a single row of $N$ nodes connected in tandem. The one-hop connectivity of the network, defined as the fraction of the $NM(NM+1)$ possible links that are operative, is simply the sum of the elements of $A_{star\text{-}mesh}$ divided by $NM(NM+1)$, or

$$\textit{Connectivity} = \frac{M \times 2(N-1) + 2M \times N + 2M \times 1}{NM(NM+1)} = \frac{4}{NM+1} \tag{15}$$

which is twice the connectivity as that for the star network with the same number of nodes.

With some observation, it can be verified that the (multihop) connectivity matrix for the star-mesh network with $M$ cross-connected rays of $N$ nodes plus a center node has the form given by

$$C_{star\text{-}mesh} = \begin{bmatrix} C_N & C_N + U_N & C_N + 2U_N & \cdots & C_N + U_N & \boldsymbol{v}_N \\ C_N + U_N & C_N & C_N + U_N & \cdots & C_N + 2U_N & \boldsymbol{v}_N \\ C_N + 2U_N & C_N + U_N & C_N & \cdots & C_N + 2U_N & \boldsymbol{v}_N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_N + U_N & C_N + 2U_N & C_N + 2U_N & \cdots & C_N & \boldsymbol{v}_N \\ \boldsymbol{v}_N^T & \boldsymbol{v}_N^T & \boldsymbol{v}_N^T & \cdots & \boldsymbol{v}_N^T & 0 \end{bmatrix} \tag{16}$$

where $C_N$ is the connectivity matrix for a row of $N$ nodes, $\boldsymbol{v}_N^T = (1,\ 2,\ 3, \cdots, N)$ is a $1 \times N$ vector (the transpose of $\boldsymbol{v}_N$), and $U_N$ is the special $N \times N$ matrix whose entries are all 1s. That is, except for the last column and last row, $C_{star\text{-}mesh}$ is an $M \times M$ matrix of $N \times N$ matrices, with the diagonal $N \times N$ matrices equal to $C_N$, the $N \times N$ matrix entries "rotationally adjacent" to the diagonals equal to $C_N + V_N$, and the other $N \times N$ matrix entries equal to $C_N + 2V_N$.

For example, the connectivity matrix for a star-mesh network with four rays of four nodes plus a center node is the following $17 \times 17$ matrix:

$$C_{star\text{-}mesh} = \begin{bmatrix}
\mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 1 \\
\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 & 2 & 1 & 2 & 3 & 2 \\
\mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 & 3 & 2 & 1 & 2 & 3 \\
\mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 4 & 3 & 2 & 1 & 4 \\
1 & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 1 \\
2 & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 & 2 \\
3 & 2 & 1 & 2 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 & 3 \\
4 & 3 & 2 & 1 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 4 \\
2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 2 & 3 & 4 & 1 \\
3 & 2 & 3 & 4 & 2 & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 & 1 & 2 & 3 & 2 \\
4 & 3 & 2 & 3 & 3 & 2 & 1 & 2 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 & 2 & 1 & 2 & 3 \\
5 & 4 & 3 & 2 & 4 & 3 & 2 & 1 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 & 3 & 2 & 1 & 4 \\
1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 \\
2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 & 2 & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 \\
3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 & 3 & 2 & 1 & 2 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 \\
4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 4 & 3 & 2 & 1 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 0
\end{bmatrix} \tag{17}$$

The maximum hop distance between node pairs is $N + 1$. The average hop distance is the sum of the elements of $C_{star}$ divided by $NM(NM + 1)$. To derive the average hop distance for the star network, we use the notation $\|X\|_+$ to denote the sum of all the elements of matrix $X$. Then the average hop distance is given by

$$\overline{m} = \frac{\|C_{star\text{-}mesh}\|_+}{NM(NM + 1)} \tag{18}$$

By inspection of (16), the sum of the elements of $C_{star}$ for $M \geq 3$ is given by

$$\|C_{star\text{-}mesh}\|_+ = M^2 \|C_N\|_+ + 2M\|\boldsymbol{v}_N\|_+ + 2M\|U_N\|_+ + M(M-3)\|2U_N\|_+ \tag{19}$$

where $\|\boldsymbol{v}_N\|_+ = \frac{1}{2}N(N+1)$, $\|nU_N\|_+ = nN^2$, and $\|C_N\|_+ = N(N-1)\left(\frac{N+1}{3}\right)$. Thus

$$\begin{aligned} \|C_{star\text{-}mesh}\|_+ &= M^2 \cdot N(N-1)\left(\tfrac{N+1}{3}\right) + MN(N+1) + 2M(M-2)N^2 \\ &= \tfrac{1}{3}NM(MN^2 + 6MN - M - 9N + 3) \end{aligned} \tag{20}$$

and the average hop distance for a star-mesh network is

$$\overline{m} = \frac{MN^2 + 6MN - M - 9N + 3}{3(MN + 1)} \tag{21}$$